# q-exponential Variable-form of a Stem Taper and Volume Model for Scots Pine (*Pinus sylvestris* L.) in Lithuania

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**Petrauskas, E., Rupšys, P. and Memgaudas, R.** 2011. q-exponential Variable-form of a Stem Taper and Volume Model for Scots Pine (*Pinus sylvestris* L.) in Lithuania. *Baltic Forestry* 17(1): 118–127.

*Abstract*

Two new taper models for stem profile (taper) were developed using q-exponential function. Five previously constructed models and two new q-exponential taper models were compared to the observed values of diameter outside bark and stem volume. Data used in this study came from stem analysis on 1,925 Scots pine trees. Results show that the q-exponential segmented taper model was superior to the Burkhart, Kozak, Lee, Sharma taper models in predicting diameter outside bark and stem volume. The results are implemented in the symbolic computational language MAPLE.

**Key words:** Stem taper model, q-exponential function, generalized least squares method

## Introduction

Taper or stem profile functions relate the diameter at any point on the stem to the height at that point. In more sophisticated stem taper models, the diameter outside or inside the bark is assumed to be a function of tree variables such as the diameter at breast height and the total tree height. Using the taper model approach, it is possible to provide predictions for outside or inside bark diameters at any point on the stem and for the stem volume from any given diameter at breast height and the total height of an individual tree (Kozak 1988). It is also possible to estimate the height for a given diameter, and by integrating the taper function between two given heights, to estimate the single-tree log volume between any two points on the stem. Taper models are often used because they are simple to analyse with associated inferential statistical methods and can give rise to explicit or implicit formulae for the diameter at any given height and volume of any merchantable segment of a single tree.

Forest researchers have shown considerable interest in taper models for improving tree volume predictions. There has been much continuing interest in the taper model, as evidenced by the continuing publication of papers in the last decades. For many years, a segmented taper model has been the commonly used approach in forestry for modelling the profile of a tree bole. The segmented taper model utilises different polynomials for the lower, middle and upper sections of the stem (Max and Burkhart 1976, Fang et al. 2000, Li and Weiskittel 2010, and references therein). The second approach, the variable-form or variable-exponent single-function taper model, was introduced by Kozak (1988). In variable-form or variable-exponent models, a simple, single, continuous function describes the diameter changes from the ground to the top of the tree (Kozak 2004, Sharma and Zhang 2004, Sharma and Parton 2009, and references therein).

The majority of taper models investigated to date are species-specific, and all the mathematical expressions in these taper models are empirical equations. The unsatisfactory predictions of the stem diameter outside or inside the bark at any given height with a minimum variance have led to the development of more advanced taper functions (Westfall and Scott 2010). Within this context, the objective of this paper was to develop a q-exponential shape taper equation that would apply over larger areas of pine stands. We know that tree growth is sigmoidal (Vanclay 1994), and several sigmoidal growth models have been proposed to describe a simple, single, continuous taper function, such as the Mitcherlich, Verhulst and Gompertz model (Sweda 1988). Recently, the generalisations of the exponential and logarithmic functions have attracted the attention of many researchers (Tsallis 2004). Using a q-exponential function, we propose the variable-form, simple, single, continuous function model and a segmented taper model that consists of q-exponential and polynomial functions.

Scots pine tree stands are dominant in Lithuanian forests. They cover 35% of the total stand areas. Since 1983, stem form-factor models created by Kuliešis et al. (1983) have been used for stem volume calculations. Stem form-factor is expressed as a function of the stem diameter at breast height and the total tree height in the A. Kuliešis model. The stem taper models that are presented in this paper increased the stem volume assessment accuracy and the applicability of the results. They could be used for forecasting structure of assorted timber during forest inventories.

Our main contribution is to expand taper models by using q-exponential functions and to compare the taper equations' performance in predicting the diameter outside the bark at any given height and stem volume. In light of recent findings (Tsallis 2004), we discuss the choice of q-exponential function for modelling stem profiles and stem volumes.

## Materials and methods

### *Data*

A total of 1925 Scots pine tree stem measurements were used for stem taper model analysis. All of the data were collected during the period of 1979–2008, and the data cover the whole Lithuania except the Curonian Spit.

Temporary circle sample plots with a 15-meter radius were placed in randomly selected clear cutting areas within each of the 42 Lithuanian state forest enterprises. All of the trees in each sample plot were felled. Only Scots pine trees were measured in this study. The diameters over the bark and under the bark of each stem were measured every 2 meters starting from the root collar, i.e., at 1, 1.3, 3, 5 m, etc. In total, 24,658 measurements were taken. The diameter was measured with a precision of 1 mm. The largest sections of sample plots with Scots pine trees were located on *Arenosols* and *Podzols* soil site types. The age of the trees varied between 19 and 164 years. No silvicultural treatments during the last ten-year period were carried out on the stands used for the study with the exception of the salvage cutting of dead trees.

The summary statistics for the diameter at breast height (D), the total height (H) and the age of all trees used for fitting and comparing stem forms are presented in Table 1. The relative diameters as a function of the relative heights are presented in Figure 1.

**Table 1.** Summary statistics of the data used for fitting stem forms

| Data | Number of trees | Min | Max | Mean | Standard Deviation |
|---|---|---|---|---|---|
| D (cm) | 1925 | 3.8 | 62.8 | 26.5 | 10.06 |
| H (m) | 1925 | 3.8 | 35.2 | 21.7 | 5.13 |
| A (yr) | 1925 | 19 | 164 | 80.4 | 25.94 |

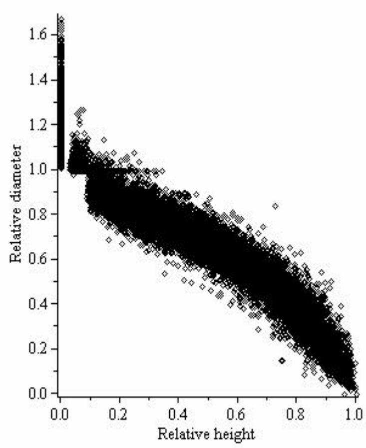



**Figure 1.** Scatter plots of the relative diameter (diameter outside bark/diameter outside bark at breast height) against the relative height (height/total height)

### *Taper equations*

As pointed out by Li and Weiskittel (2010), Rojo et al. (2005) and other researchers, the most commonly used taper models were developed by Max and Burkhart (1976) and Kozak (1988). Based on these investigations, in this study, five known taper models and two newly developed models were utilised for evaluation.

We assume that – $d$ is the diameter outside the bark at any given height $h$, $D$ is the diameter at breast height outside the bark, $H$ is the total tree height from ground to tip and $z=\frac{h}{H}$.

***Model 1.*** Max and Burkhart's (1976) segmented polynomial model is written as follows:

$$\frac{d^2}{D^2}=\beta_1(z-1)+\beta_2(z^2-1)+\beta_3(\alpha_1-z)^2I_1(\alpha_1-z)+\beta_4(\alpha_2-z)^2I_2(\alpha_2-z) \quad (1)$$

where $\alpha_1$ and $\alpha_2$ are the segmented joint points of the tree segments, $\beta_1$–$\beta_4$ are the parameters to be estimated from data and

$$I_i(\alpha_i-z)=\begin{cases}1 \; if \; \alpha_i-z\geq 0,\\ 0 \; otherwise\end{cases}, \; i=1,2$$

A few different formulations of taper models were proposed by Kozak (1988, 2004). The two variable-exponent, single continuous-function taper models are given below:

***Model 2.***

$$d=\beta_1 D^{\beta_2} X^{\beta_3+\beta_4\exp\left(-D/H\right)+\beta_5 D^X+\beta_6 X^{D/H}} \quad (2)$$

where

$X=\frac{1-\left(h/H\right)^{1/4}}{1-(0.01)^{1/4}}$, $\beta_1$–$\beta_6$ are the parameters to be estimated from data.

***Model 3.***

$$d = \beta_1 D^{\beta_2} H^{\beta_3} X^{\beta_4 z^4 + \beta_5 \exp\left(-D/H\right) + \beta_6 X^{0.1} + \beta_7 D^{-1} + \beta_8 H^Q + \beta_9 X} \tag{3}$$

where

$X = \frac{\left(1 - h/H\right)^{1/3}}{(1-p)^{1/3}}$, $Q = 1 - z^{1/3}$, $p = 1.3/H$, $\beta_1$–$\beta_9$ are the parameters to be estimated from data.

***Model 4.*** Lee et al.'s (2003) variable-exponent, single continuous-function taper model is defined as follows:

$$d = \beta_1 D^{\beta_2} (1-z)^{\beta_3 z^2 + \beta_4 z + \beta_5}, \quad \beta_1,\ \beta_2,\ \beta_3,\ \beta_5 > 0,\ \beta_4 < 0 \tag{4}$$

where $\beta_1$–$\beta_5$ are the parameters to be estimated from data.

***Model 5.*** Sharma and Parton's (2009) variable-exponent, single continuous-function taper model is defined as follows:

$$\frac{d}{D} = \beta_1 \left(\frac{H-h}{H-1.37}\right)\left(\frac{H}{1.37}\right)^{\beta_2 + \beta_3 z + \beta_4 z^2} \tag{5}$$

where $\beta_1$–$\beta_4$ are the parameters to be estimated from data.

The advantage of generalising the exponential functions has captured the attention of researchers in recent years (Tsallis 2004). The q-generalisation enables us to stress symmetry properties of the growth process. The solution of equation $\frac{dy}{dx} = y^q$ (y(0) =1) is given by a q-exponential function:

$$y \equiv e_q^x = \begin{cases} \left[1 - (1-q)x\right]^{\frac{1}{1-q}} & if\ 1 - (1-q)x \ge 0, \\ 0 & otherwise \end{cases}$$

The more general growth process takes the following form:

$$\frac{dy}{dx} = \alpha y + \beta y^q$$

whose solution is defined by a q-exponential function as follows:

$$y = \begin{cases} \left[\delta - \frac{\beta}{\alpha}\left(1 - \exp\left((1-q)\alpha x\right)\right)\right]^{\frac{1}{1-q}} & if\ \delta - \frac{\beta}{\alpha}\left(1 - \exp\left((1-q)\alpha x\right)\right) \ge 0, \\ 0 & otherwise \end{cases} \tag{6}$$

***Model 6.*** Using q-exponential function (6), a variable-form, single continuous-function taper model was defined in the following form:

$$d = \beta_1 D^{\beta_2} \begin{cases} \left[\beta_3 - \frac{\beta_4}{\beta_5}\left(1 - \exp\left((1-\beta_6)\beta_5 z\right)\right)\right]^{\frac{1}{1-\beta_6}} & if\ \beta_3 - \frac{\beta_4}{\beta_5}\left(1 - \exp\left((1-\beta_6)\beta_5 z\right)\right) \ge 0, \\ 0 & otherwise \end{cases} \tag{7}$$

where $\beta_1$–$\beta_6$ are the parameters to be estimated from data.

***Model 7.*** Using q-exponential function (6) and a parabola, a segmented taper model was defined in the following form:

$$d = \beta_1 D^{\beta_2} \begin{cases} \left[\beta_3 - \frac{\beta_4}{\beta_5}\left(1 - \exp\left((1-\beta_6)\beta_5 z\right)\right)\right]^{\frac{1}{1-\beta_6}} & if\ z \le a, \\ \beta_7 + \beta_8 z + \beta_9 z^2 & otherwise \end{cases} \tag{8}$$

where $\beta_1$–$\beta_9$ are the parameters to be estimated from data and $a$ is the segmented joint point of the q-exponential function and the parabola.

***Stem volume equations***

Most applications of taper equations are for stem volume predictions. The main advantage in using taper curves is that if the tree profile can be accurately described, the volume for any merchantable segment can be computed by integrating the equation. The taper equation for stem diameter, *d=d(D,H,h)*, allows us to revise the prediction of the total stem volume, *V(D,H)*, into the following form:

$$V(D,H) = \frac{\pi}{40000}\int_0^H d^2(D,H,h)\cdot dh \tag{9}$$

In this study, the volume predictions defined by Eq. (9) were also compared with the Honer (1965) volume equation and the Kuliešis et al. (1983) volume equation defined by a stem form-factor equation. The Honer volume model does not require the use of a taper equation, and the estimation of the stem volume is based on a statistical regression technique that directly relates the volume to such quantities as the diameter outside the bark at breast height and the total height.

***Model 8.*** (Honer).

$$V(D,H) = \frac{D^2}{\beta_1 + \beta_2/H} \tag{10}$$

where $\beta_1$–$\beta_2$ are the parameters to be estimated from data.

The Kuliešis et al. (1983) model for estimating the stem volume is based on a form-factor equation that is derived from the dependence of the volume on the diameter outside the bark at breast height and on the total height.

***Model 9.*** (Kuliešis et al.).

$$V = \frac{\pi D^2 H}{40000}\cdot F(D,H), \tag{11}$$

where $F(D,H) = \beta_1 + \frac{\beta_2}{H} + \frac{\beta_3}{D} + \frac{\beta_4}{H\cdot D} + \frac{\beta_5}{D^2} + \frac{\beta_6}{D^2\cdot H}$, $\beta_1 - \beta_6$ are the parameters to be estimated from data.

Using the observed diameters for each tree section between two adjacent diameter measurements, the volume was calculated. The volume of the top section was assumed to be conic in shape, and the volumes of all other sections were assumed to be shaped like segmented cones. Hence, Smalian's formula for the *i-th* tree is defined as follows:

$$V_i = \frac{\pi}{40000}\left(\sum_{k=1}^{n_i-2}\frac{\left(d_{ik}^2 + d_{ik+1}^2\right)\cdot L_{ik}}{2} + \frac{d_{in_i-1}^2 \cdot L_{in_i-1}}{3}\right)$$

where $d_{ik}^2$ is the diameter (cm) for section $k$ of tree $i$ and $L_{ik}$ is the section length (m). Unfortunately, the rate of tree taper from the base to the tip is not uniform throughout the stem. The greater the difference between the two adjacent diameter measurements, the less reliable the volumes obtained using Smalian's formula will be. The volume of the tree stem sections can also be derived using a truncated cone formula:

$$V_i = \frac{\pi}{3\cdot 40000}\left(\sum_{k=1}^{n_i-2}\left(d_{ik}^2 + d_{ik+1}^2 + d_{ik}\cdot d_{ik+1}\right)\cdot L_{ik} + d_{in_i-1}^2 \cdot L_{in_i-1}\right) \quad (12)$$

In this paper, the volume defined by Eq. (12) was considered to be the observed volume.

***Statistical analysis***

The construction of taper equations requires the collection of hierarchical measurements on individual trees. These longitudinal data have two characteristics that complicate their statistical analysis: a) within-tree correlations that appear with data measured on the same tree and b) independence but extremely high variability between the experimental curves of the different trees. The poor fit of the taper models is clearly illustrated by the existence of dispersion clusters of highly correlated residuals corresponding to the data coming from each individual tree. Therefore, in taper models, autocorrelation may be thought of as representing the situation in which close observations in two adjacent diameter measurements may be correlated, but correlations among those spaced farther apart are negligible. For taper model fitting to improve the estimation efficiency, we used a generalised least squares technique to account for the correlations among the data. The correlation structure is defined by the first-order autoregressive process AR (1), and it is assumed that $\rho=\rho_1$.

Recently, taper models were investigated using a prediction of the random effects based on supplementary diameter measurements (Sharma and Parton 2009, Li and Weiskittel 2010, and references therein). More appropriate methods for incorporating the structured variance in the prediction exist and need to be explored by means of stochastic differential equations (Rupšys et al. 2007, Rupšys and Petrauskas 2010a, 2010b).

The seven taper models were compared with the observed values of the diameter outside the bark and the stem volume. Numerical and graphical analyses of the residuals were used as criteria for comparing the taper equations. The performance statistics of the taper equations for the diameter and the volume included five fit statistics: mean bias (B), mean absolute bias (MAB), the mean percentage of absolute bias (MPB), relative error (RE%), and the coefficient of determination ($R^2$):

$$B = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right) \qquad MAB = \frac{1}{n}\sum_{i=1}^{n}\left|y_i - \hat{y}_i\right| \qquad MPB = \frac{\sum_{i=1}^{n}\left|y_i - \hat{y}_i\right|}{\sum_{i=1}^{n} y_i} * 100$$

$$RE\% = \frac{\sqrt{\frac{1}{n-p}\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}}{\bar{y}} \qquad R^2 = 1 - \frac{\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}$$

where $n = \sum_{i=1}^{m} n_i$ is the total number of observations used to fit the model, $m$ is the number of trees; $y_i$, $\hat{y}_i$ and $\bar{y}$ are the measured, estimated and average values of the dependent variable (diameter outside the bark or the stem volume), and $p$ is the number of model parameters.

All results are implemented in the symbolic algebra system MAPLE using the **Statistics** and **MultivariateCalculus** libraries.

## Results

The parameters for all of the examined taper models for the data set summarised in Table 1 and Figure 1 are presented in Table 2. As we can see, all parameters are highly significant ($\alpha$=0.05), with the exception of Kozak model (2) (the parameter at $D^x$) and Kozak model (3) (the parameter at $D^{-1}$). As shown by Kozak (2004), the variable-exponent models (2) and (3) have a high multicollinearity problem. As shown by Sharma and Burkhart (2003), the fit statistics of the segmented Burkhart's taper model are optimised when the lower and upper joint points for total tree height range from 6% to 15% and from 60% to 85%, respectively. For the segmented Burkhart's taper model, the lower and upper joint points that were adopted were 0.11 and 0.75, respectively. For the q-exponential segmented taper model defined by Equation (8), a joint point was taken at 0.5 because the fit statistics produced the best values. The fit statistics for the diameter outside the bark for the seven taper models are presented in Table 3.

As seen in Table 3, all of the taper models account for at least 97% of the variation in the diameter outside the bark. The mean bias and the mean percentage of bias for models (3), (4) and (8) were negative, indicating that these models are slightly overpredicting by 0.40%, 0.13% and 0.04%, respectively. For four of the models—(1), (2), (5) and (7)—the mean bias and the mean percentage of bias were positive, indicating that these models are slightly underpredicting (from 0.23% to 0.52%). These values are generally small, and

**Table 2.** Estimated parameters (standard errors in parentheses) of the seven taper models

| Model | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ |
|---|---|---|---|---|---|---|---|---|---|
| Eq.(1) | -0.3872 | 1.3803 | 49.6334 | -1.5728 | - | - | - | - | - |
| | (0.0042) | (0.0141) | (0.2271) | (0.0195) | | | | | |
| Eq. 2) | 1.4009 | 0.9315 | 0.4527 | -0.1059 | 0.00009 | -0.1334 | - | - | - |
| | (0.0091) | (0.0020) | (0.0029) | (0.0101) | (0.00005) | (0.0043) | | | |
| Eq.(3) | 1.0156 | 0.9289 | 0.0709 | 0.4449 | -0.2112 | 0.4232 | -0.0269 | -0.0052 | 0.0942 |
| | (0.0084) | (0.0015) | (0.0023) | (0.0034) | (0.0098) | (0.0029) | (0.0905) | (0.0003) | (0.0040) |
| Eq.(4) | 1.6102 | 0.9025 | 2.7407 | -3.8678 | 2.0559 | - | - | - | - |
| | (0.0079) | (0.0014) | (0.0275) | (0.0.48) | (0.0114) | | | | |
| Eq.(5) | 1.2278 | -0.0523 | -0.1607 | 0.4935 | - | - | - | - | - |
| | (0.0051) | (0.0015) | (0.0015) | (0.0021) | | | | | |
| Eq.(7) | 2.2194 | 0.9085 | 0.3345 | -3.0899 | -0.8097 | -1.7018 | - | - | - |
| | (0.0017) | (0.0196) | (0.0129) | (0.0088) | (0.0328) | (0.1016) | | | |
| Eq.(8) | 1.2682 | 0.9291 | 0.0693 | -1.0134 | -0.7453 | 13.9083 | 0.3813 | 1.4481 | -1.8234 |
| | (0.0055) | (0.0012) | (0.0073) | (0.0433) | (0.0144) | (0.5001) | (0.0214) | (0.0562) | (0.0358) |

**Table 3.** Mean bias (B, cm), mean absolute bias (MAB, cm), mean percentage of absolute bias (MPB, %), relative error (RE%) and coefficient of determination ($R^2$) of diameter outside bark

| Model | B, cm (B%*) | MAB, cm | MPB, % | RE% | $R^2$ |
|---|---|---|---|---|---|
| Eq. (1) | 0.0915 (0.52) | 1.0793 | 6.10 | 8.64 | 0.9812 |
| Eq. (2) | 0.0770 (0.44) | 1.2178 | 6.87 | 9.46 | 0.9775 |
| Eq. (3) | -0.1052 (-0.40) | 0.9757 | 5.51 | 7.84 | 0.9845 |
| Eq. (4) | -0.0234 (-0.13) | 1.1182 | 6.32 | 8.84 | 0.9803 |
| Eq. (5) | 0.0428 (0.24) | 1.2387 | 7.00 | 9.81 | 0.9758 |
| Eq. (7) | 0.0414 (0.23) | 1.3183 | 7.44 | 10.75 | 0.9709 |
| **Eq. (8)** | **-0.0063 (-0.04)** | **0.9407** | **5.31** | **7.80** | **0.9846** |

* $B\% = \frac{B}{y} * 100$. The lowest values of the first four fit statistics and the largest value of the coefficient of determination for all taper models are in bold

the models are overall unbiased. The MAB of all models ranged from 0.94 to 1.32 cm (MPB range: from 5.31% to 7.44%). The relative error of prediction ranged from 7.80% to 10.75%. For prediction of the diameter outside the bark at any point on the stem, the q-exponential segmented model (8) was superior to commonly used taper models. Overall, for modelling the diameter outside the bark, the best-fit statistics showed the Burkhart, the Kozak and the q-exponential segmented models defined by equations (1), (3) and (8) to be the best models.

For the three best taper models—(1), (3) and (8)—the measures of mean bias, mean percentage of absolute bias and relative error using diameter measurements for different relative heights are presented in Table 4. For the Kozak model (3), the diameters outside the bark are overpredicted in the lower portion of the bole. In terms of mean bias, mean percentage of absolute bias and relative error, the q-exponential segmented model (8) appears to be superior for the prediction of diameters outside the bark.

To illustrate how well our presented q-exponential taper models predict the stem volume, the observed (12) and predicted (9) volumes were calculated. These predictions were compared with the predictions calculated using the other taper models and the Honer and Kuliešis et al. stem volume equations. The coefficients of the Honer volume model and the Kuliešis et al. form factor equation were estimated by the ordinary least squares technique. The estimators were specified as follows: $\beta_1$ = 271.1455 and $\beta_2$ = 21672.5044 (Honer model) and $\beta_1$ = 0.3853, $\beta_2$ = 0.2478, $\beta_3$ = 2.1801, $\beta_4$ = 1.0489, $\beta_5$ = –12.4538 and $\beta_6$ = 28.6451 (Kuliešis et al. model).

The fit statistics are displayed in Table 5. Careful examination of Table 5 reveals that the stem volume predictions showed a slightly different ranking than the predictions of the diameter outside the bark. Overall, the q-exponential taper models, defined by equations (7) and (8), yielded the best mean percentage of bias, mean absolute bias and mean percentage of absolute bias. An analysis of the results in Table 5 based on the mean bias, the relative error and the coefficient of determination indicated that Lee's taper model, as defined by equation (4), was best.

**Table 4.** Mean bias (B, cm), mean percentage of absolute bias (MPB, %) and relative error (RE%) by relative heights

| Relative heights | Number Obs. | Burkhart (1) | | | Kozak (3) | | | q-exponential segmented (8) | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | B | MPB | RE% | B | MPB | RE% | B | MPB | RE% |
| **0.0≤h/H≤0.1** | 5649 | 0.2211 | 4.54 | 6.32 | -0.0709 | 3.40 | 5.49 | **0.0223** | **3.21** | **5.25** |
| **0.1<h/H≤0.2** | 2235 | 0.4285 | 4.44 | 5.38 | -0.3204 | 3.78 | **4.94** | **-0.1185** | **3.69** | 4.96 |
| **0.2<h/H≤0.3** | 1958 | -0.0735 | 4.62 | 6.25 | -0.1489 | 4.36 | 5.74 | **0.0611** | **4.28** | **5.69** |
| **0.3<h/H≤0.4** | 1989 | -0.1238 | 5.03 | 6.82 | **-0.0983** | 4.89 | 6.45 | 0.1166 | **4.78** | **6.33** |
| **0.4<h/H≤0.5** | 2003 | 0.0068 | 5.51 | 7.60 | -0.0817 | 5.41 | 7.15 | **-0.0026** | **5.22** | **7.02** |
| **0.5<h/H≤0.6** | 1968 | 0.1335 | 6.31 | 8.66 | -0.1203 | 6.11 | **8.19** | **0.0108** | **5.99** | 8.20 |
| **0.6<h/H≤0.7** | 1966 | 0.2513 | 7.96 | 10.71 | **-0.0578** | 7.37 | **10.04** | -0.1261 | **7.36** | 10.14 |
| **0.7<h/H≤0.8** | 1998 | 0.2494 | 10.85 | 14.37 | **0.0175** | **9.77** | **13.39** | -0.1068 | 10.12 | 13.91 |
| **0.8<h/H≤0.9** | 1957 | -0.0742 | 17.07 | 22.68 | 0.1240 | **15.73** | **21.15** | **0.0563** | 16.21 | 21.77 |
| **0.9<h/H≤1.0** | 2939 | -0.2329 | - | - | -0.2570 | - | - | **-0.0116** | - | - |
| **All Obs** | 24658 | 0.0915 | 6.10 | 8.64 | -0.1052 | 5.51 | 7.84 | **-0.0063** | **5.31** | **7.80** |

The lowest values of the three fit statistics for each relative height class are in bold

Figures 2 and 3 show the residuals plotted against predictions of the diameter at any given height and stem volume. Graphical diagnostics of the residuals for the diameter predictions showed that the residuals of the Burkhart (1), Kozak (3) and q-exponential (8) models had more homogeneous variance than other taper models. For volume predictions, the best fits were obtained using the Kozak (3), q-exponential (7) and q-exponential segmented (8) taper models. These results are similar to the evaluation obtained using fit statistics. However, for the predictions of stem volume, the residuals increased steadily with stem volume. The trend becomes less obvious when the predicted volume is less than 1.5 $m^3$. To further improve the predictions of gross trees, the first step could be the detection and modelling of a variance function and the addition of new predictors, such as stand density and the ratio of crown length.

**Table 5.** Mean bias (B, $m^3$), mean absolute bias (MAB, $m^3$), mean percentage of absolute bias (MPB, %), relative error (RE%) and coefficient of determination ($R^2$) of stem volume

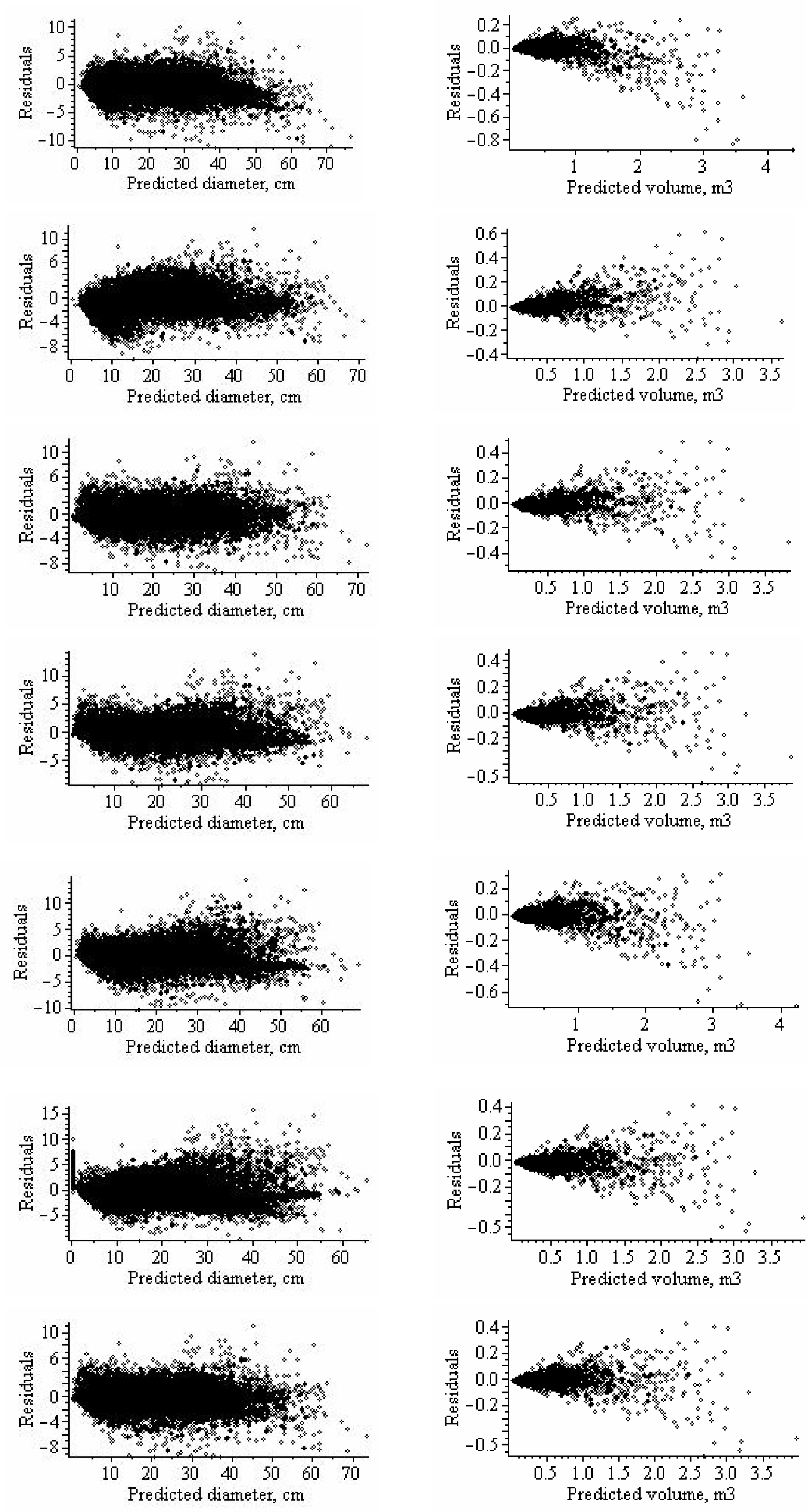
| Model | B, $m^3$ (B%)* | MAB, cm | MPB, % | RE% | $R^2$ |
|---|---|---|---|---|---|
| **Eq. (1)** | -0.0001 (4.22) | 0.0482 | 7.89 | 14.40 | 0.9760 |
| **Eq. (2)** | 0.0263 (2.07) | 0.0476 | 7.78 | 12.31 | 0.9818 |
| **Eq. (3)** | 0.0046 (-1.62) | 0.0416 | 6.80 | 11.55 | 0.9846 |
| **Eq. (4)** | **0.0006** (-2.14) | 0.0412 | 6.73 | **11.47** | **0.9848** |
| **Eq. (5)** | 0.0020 (-0.59) | 0.0442 | 7.22 | 12.92 | 0.9801 |
| **Eq. (7)** | -0.0049 (-1.98) | 0.0412 | **6.70** | 11.58 | 0.9845 |
| **Eq. (8)** | 0.0060 (0.58) | **0.0411** | 6.73 | 11.64 | 0.9843 |
| **Eq. (10)** | 0.0077 (1.64) | 0.0443 | 7.25 | 12.50 | 0.9840 |
| **Eq. (11)** | 0,0017 (**0.31**) | 0.0416 | 6.78 | 12.08 | 0.9831 |

$^*B\% = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i - \hat{y}_i}{y_i}\right)*100$. The lowest values of the first four fit statistics and the largest value of the coefficient of determination for all taper models are in bold

Taper profiles for three randomly selected Scots pine trees with diameters outside the bark at breast height of 19.6, 38.2 and 45.2 cm and total tree heights of 19.1, 22.4 and 31.9 m, respectively, using three different taper equations (Eq. (3), Eq. (7) and Eq. (8)), are plotted in Figure 4. It is clear that all tree profiles followed the stem data very closely. A graphical examination leads to the conclusion that the q-exponential segmented model (8) describes the stem taper quite well and is comparatively superior to commonly used taper models.

## Discussion

The importance of form-factors and taper models is their applicability to evaluate volume of full length or merchantable part of stem, apparent log volume and stem surface area (Inoue 2006). Although there have been numerous studies there is still no consensus on whether stem taper which is inherent to tree species depends on growing conditions such as geographical location, soil type, stand density, genetic features etc., and whether the resultant differences in stem taper are significant enough to be described employing statistical models. A number of various taper models were created during the last several decades. Stem taper was




Burkhart model (1)

Kozak model (2)

Kozak model (3)

Lee model (4)

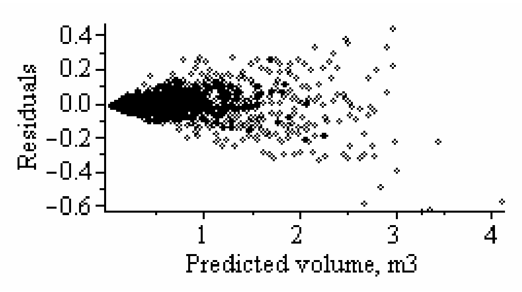
Sharma model (5)

q-exponential model (7)

q-exponential segmented model (8)

**Figure 2.** Plot of residuals versus predicted diameter (left) and predicted stem volume (right)

Honer model




Kuliešis et al. model

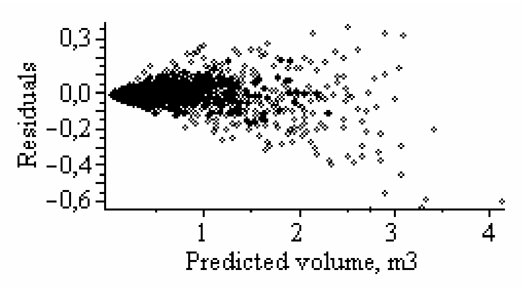



**Figure 3.** Plot of residuals versus predicted stem volume: the Honer model (top), the Kuliešis et al. model (bottom)

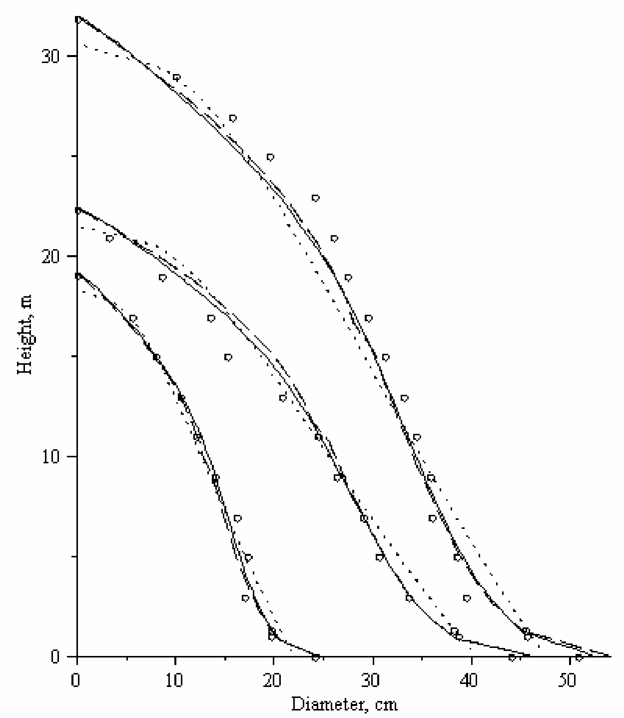



**Figure 4.** Tree profiles for three randomly selected pine trees generated using Kozak (3) (solid line), q-exponential (7) (dot line) and q-exponential (8) (dash line) taper models

approximated by means of single function, in order to find the best fit function type or parameter estimation method (Kozak 1988, 2004, Lee et al. 2003, Sharma and Parton 2009). Segmented taper models were introduced (Max and Burkhart 1976) taking into account a proven fact that different parts of the stem have sigmoid, parabolic and cone shapes. Segmented function taper models have fixed (Max and Burkhart 1976) or estimated joint points (Westfall and Scott 2010). In principle all those models were dedicated for developing universal taper function per tree species. Kuliešis (1972) undertook the first attempt to pick out effects of different factors influencing stem taper function using multi-factorial ANOVA. He found that stem form coefficients $q_3$, $q_5$, $q_7$ ($d_{0..3}/d_{0.1}$; $d_{0..5}/d_{0.1}$; $d_{0..7}/d_{0.1}$) at relative heights 0.3, 0.5, 0.7 are not stable but significantly depend on tree growing conditions, total height of tree, DBH, and the length of crown. It was highlighted that trees of one species have no permanent stem taper curve. The recent studies analyse applicability of mixed effects methods to estimate parameters of stem taper function incorporating extra measurable parameters of individual trees (Li and Weiskittel 2010, Westfall and Scott 2010). Those parameters in case of each individual stem could describe taper deviation from a universal taper which should be typical of tree species. Such a type of models is usually local and of restricted applicability. Difficulties arise in selecting tree parameters, which could properly reveal complex of changes in tree growing conditions. Partial change in growing conditions may not present singularities of stem taper and lead to significant bias in forecasts. The main aim of investigations at this stage was to create stem taper function for Scots pine growing on Lithuanian territory, which would allow calculating volume of all stems (alternative to Kuliešis (1983) form factor model) as well as of any part of the stem, which would be used for assessment of different round wood assortments. The next stage of investigations will be increasing stem taper function flexibility in order to describe stem taper profile taking into consideration tree growing conditions.

Many stem form models using theoretical sigmoidal functions have been proposed. Thus the sigmoidal taper curves provide a quantitative description of stem form. As was discussed by Sweda (1988), none of the Mitcherlich, Verhulst and Gompertz taper curves is flawless. Some of them appear to be defected only at the bases of a tree and other at the both bases and apexes. In regard to this, we have developed the possible generalisations of logistic shape taper curve on the basis of a recently introduced one-parameter exponential function. The one-parameter generalisations of the exponential function are applied in a wide range of disciplines since they permit to improve logistic models. The stem taper models developed represent a q-exponential version of the relative height variable equation. They have six or nine (segmented form) general parameters when the outside bark diameter is treated as the dependent variable. The use of a generalised least squares technique with a first-order autoregressive process AR(1) structure represents an advantage over the ordinary least square estimators. It allows efficient estimation of the parameters of the models if the autocorrelative pattern for observations coming from the same tree is adequate. Autocorrelation is a common matter of investigation associated with data used to fit taper models. In terms of accuracy, the proposed models show biases that are similar in range to those of other studies. The mean percentage of absolute bias and the relative error by the relative heights in the upper sections are particularly large. As was shown in Lejeune et al. (2009), the low reliability in the upper bole section can be explained by the cylindrical assumption and the lack of diameter measurements in the upper bole sections.

The volume-diameter-height relationships of a tree species are highly site-dependent and non-constant over time. Incorporating datasets from different plots into a single comprehensive analysis introduces some inhomogeneity and affects the resulting predictions. To account for stand-level variation, Sharma and Zhang (2004) generalised these relationships using stand characteristics. Comparing several models, they concluded that the generalised models were similar in terms of fit statistics. Zianis et al. (2005) compiled 230 stem volume equations, and they covered 55 species altogether. In the majority of the compiled stem volume models, the independent variables were diameter at breast height and/or height. The stem volume models estimated using the developed q-exponential stem tapers show fit statistics that are similar in range to those of other studies.

## Conclusions

Two new taper models were developed using the q-exponential function. The gradually changing stem profiles of the tree bole from ground to tip can be expressed in terms of the q-exponential function of the relative height. Seven taper equations were fitted for Scots pine trees using a nonlinear generalised least squares technique, which were then compared using fit statistics: mean bias, mean percentage of bias, mean absolute bias, mean percentage of absolute bias, relative error, and coefficient of determination. The results showed that for modelling the diameter outside the bark, the Burkhart model (1), the Kozak model (3) and the q-exponential model (8) showed the best fit. For modelling the stem volume, the Kozak model (3), the Lee model (4), the q-exponential model (7) and the q-exponential segmented model (8) showed the best fit. Generally, for modelling the diameter outside the bark and stem volume, the best-fit statistics and the distribution of residuals showed that the q-exponential models defined by equations (7) and (8) are the best models.

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Received 29 September 2010
Accepted 05 April 2011

## ПРИМЕНЕНИЯ q-ЭКСПОНЕНЦИАЛЬНОЙ ФУНКЦИИ ДЛЯ МОДЕЛЕЙ ОБРАЗУЮЩЕЙ СТВОЛА СОСНЫ ОБЫКНОВЕННОЙ (*Pinus sylvestris* L.)

**Э. Пятраускас, П. Рупшис, Р. Мемгаудас**

*Резюме*

Две новые модели образующих стволов были разработаны с использованием q-экспоненты. Было проверено всего 7 моделей образующих стволов с эмпирическими данными диаметра ствола с корой и объема. В исследовании были использованы данные 1925 стволов сосен. Результаты показали, что сегментированная q-экспоненциальная модель образующей стволов по статистическим критериям не уступает кривым моделей образующих стволов Бурхарта и Козака. Для обработки эмпирических данных использована система MAPLE 11.

**Ключевые слова:** модели образующей ствола, q-экспоненциальная функция, обобщенный метод наименьших квадратов